J. Fluid Mech. (1975), *vol.* 71, *Part* 3, *pp.* 529–540
Printed in Great Britain

Circulations and density distributions in a deep, strongly stratified, two-layer estuary

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(Received 31 January 1975)

The paper discusses a theoretical model of statistically steady flow in a strongly stratified estuary. A halocline is assumed to be present and the lower layer is taken to be deep and non-turbulent. The outflowing upper fluid mixes with the salty lower fluid and the flux of the brackish water increases with distance from the head of the estuary. The mixing is assumed to be similar to that in laboratory models of mixing across density interfaces.

Two equations of mass conservation are used, one for the steady-state mass flux across a vertical section from top to bottom of the channel and one for the mass flux into a section of the upper fluid. A buoyancy conservation equation is used for the buoyancy flux across a vertical section. A final equation is obtained by integrating the horizontal equation of motion across a section of the upper fluid. The flow in this layer is assumed to be opposed by a frictional force proportional to the square of the velocity averaged over the layer. The pressure-gradient force arising from the slope of the free surface is solved for in terms of the thickness of the upper layer, the buoyancy difference across the interface, the slope of the interface and the horizontal density gradient in the upper layer. The derivation shows that the horizontal pressure-gradient force vanishes in the lower layer.

The mathematical problem reduces to two ordinary differential equations for the flux in the upper layer and its thickness. Attention is confined to the solution for subcritical flow, in which the interface falls with distance from the head, reaching a maximum depth at a certain section of the estuary. Beyond this the interface rises. At the mouth, where, by definition, the width of the estuary increases rapidly, it is shown that there must be a transition from subcritical to supercritical flow. This condition, applied to the solution for uniform width, determines a remaining unknown related to the depth of the halocline at the head of the estuary and the complete solution is obtained as a function of the fresh-water influx per unit width, the r.m.s. turbulent velocity, the estuary length and the buoyancy of sea water.

The solution is complicated but has reasonable behaviour for variations of the given parameters of the problem. A basic feature for values of the constants appropriate to fjord-type estuaries is the dominance of friction, omitted in an earlier, incomplete investigation by Stommel. This is also revealed by the large drop in the free surface over the length of the estuary.

A comparison with two estuaries, Oslofjord and Knight Inlet, British Columbia, indicates that the former is very different from the model of this paper but that the latter may have a similar nature.

1. Theoretical model

The circulation of the water and the distribution of density in an estuary are controlled by a number of factors, including the intensity, length scale and horizontal distribution of the turbulence, the fresh-water inflow, the imposed density differences and the geometry (Pritchard 1952; Rattray & Hansen 1962; Dyer 1973). In this paper we present a discussion of a statistically steady estuary in which there is an upper layer well-mixed vertically by turbulence and a deep non-turbulent lower layer of uniform density ρ_0. Temperature effects are neglected, so that all density variations are assumed to be caused by salinity differences. We also neglect Coriolis forces.

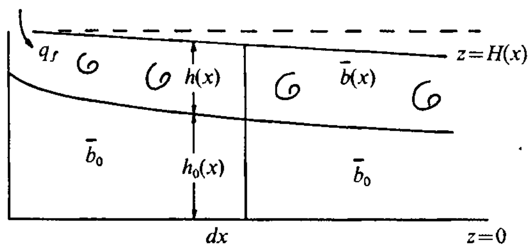

FIGURE 1. Estuary model.

Features of the idealized model are shown in figure 1. The width of the channel is $W(x)$ and the sides are vertical and equidistant from the plane $y = 0$. The bottom is the plane $z = 0$, but the results are unaffected by an irregular bottom. The lower fluid has a thickness $h_0(x)$, the upper fluid has a thickness $h(x)$ and the free surface has a height $H = h_0+h$. The variation of H along the estuary provides an important component of the pressure-gradient force, but increments in H are of the order of 10 cm over tens of kilometres and so may be neglected elsewhere in the argument. The vertical variation of the buoyancy $b = (\rho-\rho_f)g/\rho_f$ is confined to a thin interfacial zone of thickness δ, but the mean buoyancy varies continuously with x in the upper layer. In the definition of b, ρ is the density and ρ_f is the density of the fresh water. In general, a symbol such as b denotes an ensemble average or an average over a long time, and $\bar{b}$ denotes an average of b over the vertical cross-section of the upper fluid.

The buoyancy difference between the two layers is Δb and varies along the channel. The r.m.s. turbulent velocity in the upper layer is σ and we assume that σ is uniform in this layer. The non-dimensional quantity $Ri = h\Delta b/\sigma^2$ has the form of a Richardson number. If we take as typical values $h = 10^3$ cm, $\Delta b = 25$ cm s^{-2} and $\sigma = 10$ cm s^{-1}, we get $Ri = 250$. It is reasonable then to assume that the estuary is strongly stratified in the sense that $Ri \gg 1$.

Conservation of mass for a region of the upper layer of length Δx, width W and height h yields

$$\Delta(Wh\bar{u}) = \int\int v_n da, \tag{1}$$

where $\bar{u}$ is the mean velocity over the cross-section of the upper layer, da is an element of area of the interface and the integral is over the interfacial boundary of the region. The normal velocity v_n into the layer is zero when there is no mixing. We assume, however, that turbulence caused by flow of the brackish water, tidal currents and wind stresses exists in the upper layer. As indicated by many laboratory experiments (Rouse & Dodu 1955; Turner 1968; Kato & Phillips 1969; Moore & Long 1971), the turbulence will tend to move the interface downwards at a speed u_e, called the entrainment velocity, and in the steady-state model of the present paper, this must be opposed by an equal upward velocity at the interface. According to Long (1975*a*), we may take the entrainment velocity to be

$$u_e = K_n \sigma^3 / h\Delta b, \tag{2}$$

where $\bar{b}(x)$ is the buoyancy in the upper fluid, $\bar{b}_0$ is the uniform buoyancy in the lower fluid, $\Delta b = \bar{b}_0 - \bar{b}(x)$ and K_n is a constant. Let us obtain an estimate for K_n. In stable conditions (Long 1975*a*), the ratio $ql/\sigma^3 \cong \frac{1}{150}$, where q is the vertical flux of buoyancy near the interface and l is the integral length scale of the turbulence. Using $l \cong \frac{1}{14}h$, as in turbulent flow in pipes (Schlichting 1955, p. 408), we get $q \cong \sigma^3/10h$. We may relate the flux to the entrainment velocity by considering a portion of the thin interfacial layer of small length, of thickness δ and width W and integrating the equation for buoyancy conservation over the region. We get

$$-w_i \Delta b \cong q,$$

where w_i is the mean normal velocity and where terms proportional to δ are omitted. We have, of course, retained the increments in buoyancy and buoyancy flux across the thin interface, but we have taken the normal velocity to be continuous across the interface in accordance with the equation of continuity. Since the interfacial slope is very small, w_i is very close to the mean vertical velocity and therefore to u_e. We obtain $q \cong u_e \Delta b$ and $u_e \cong \sigma^3/10h\Delta b$, so that $K_n = 0{\cdot}1$ in (2). The continuity equation for the upper layer becomes

$$d(Wh\bar{u})/dx = K_n \sigma^3 W / h\Delta b. \tag{3}$$

Let us now integrate the horizontal equation of motion over a region of width W, height H and length Δx. In this operation, we may replace the density by ρ_f through the use of the Boussinesq approximation. We get

$$\iiint (\nabla . \mathbf{v}u)\, dV = -\iiint \frac{1}{\rho_f}\frac{\partial p}{\partial x}\, dV + \iiint \nabla . \tau\, dV, \tag{4}$$

where p is the pressure, $\mathbf{v}$ is the velocity, $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$ is the stress force in the x direction and dV is an element of volume. The components of $\boldsymbol{\tau}$ are nearly equal to the Reynolds stresses in the interior of the fluid. Equation (4) may be written as

$$\frac{1}{HW}\frac{d}{dx}\int_0^H \int_{-\frac{1}{2}W}^{\frac{1}{2}W} uu\, dz\, dy = -gH_x - \Delta b \frac{h_0}{H}\frac{dh_0}{dx} - \tfrac{1}{2}h\left(1 + \frac{h_0}{H}\right)\frac{d\bar{b}}{dx} - \frac{\tau_0}{H} - \frac{1}{HW}\int_0^H \tau_s\, dz, \tag{5}$$

where we have used the Boussinesq approximation and the hydrostatic relation

$$\frac{p}{\rho_f} = g(H-z) + \begin{cases} \bar{b}(H-z) & (z > h_0), \\ \bar{b}(H-h_0) + \bar{b}_0(h_0 - z) & (z < h_0), \end{cases}$$

τ_0 is the stress at the bottom boundary averaged across the estuary and τ_s is the sum of the stresses at the side walls. We neglect $\partial\tau_x/\partial x$ because we have assumed that the turbulence is homogeneous in the layer. At this step in the analysis, it is possible to include a term arising from wind stresses at the free surface. In the present paper, if a wind stress exists (and is partly responsible for the turbulence), we assume that it has a zero average over our averaging time period.

The integral on the left-hand side of (5) remains finite as $H \to \infty$, so that the left-hand side of this equation becomes negligible for a very deep lower fluid. On the right-hand side of (5), $\tau_0/H \to 0$ and the integral of the wall stress remains finite, so that the last two terms in (5) are also negligible. We obtain

$$gH_x = \Delta b dh/dx - h d\bar{b}/dx. \tag{6}$$

It follows, of course, that the pressure-gradient force along the estuary vanishes at all levels below the interface. This behaviour is commonly assumed for deep fjords (Gade 1974, unpublished manuscript).

Let us now integrate the horizontal equation of motion over a region of the upper fluid of length Δx, width W and height h. We get

$$\frac{1}{W\Delta x}\iiint \nabla . \mathbf{v} u dV = -h\Delta b\frac{dh}{dx} + h^2\frac{d\bar{b}}{dx} - \tau_i - \frac{1}{W}\int_{h_0}^{H}\int_{-\frac{1}{2}W}^{\frac{1}{2}W}\left[\int_z^H b_x dz\right] dz dy, \tag{7}$$

where τ_i is very nearly equal to the average stress at the interface. We assume $\tau_i = K\bar{u}^2$, where K is the drag coefficient. A term involving the horizontal rate of change of τ_x has been omitted in (7) because the turbulence is assumed to be uniform in the upper layer. If u_i is the average horizontal velocity at the bottom of the upper layer, the integral on the left-hand side of (7) may be written as

$$\frac{1}{W}\frac{d}{dx}\int_{h_0}^{H}\int_{-\frac{1}{2}W}^{\frac{1}{2}W} uu dz dy - u_e u_i. \tag{8}$$

The ratio of $u_e u_i$ to the term τ_i in (7) is of order $u_e h/K_m$, or less, where K_m is the eddy viscosity in the upper fluid near the interface. This ratio is small because turbulent velocities are very much larger than entrainment velocities. If we again use $l \simeq \frac{1}{14}h$, the ratio is approximately $\sigma^2/h\Delta b$, which we have assumed to be small. Thus the second term in (8) may be neglected.

Let us now write

$$\int_{h_0}^{H}\int_{-\frac{1}{2}W}^{\frac{1}{2}W} uu dz dy = \gamma\bar{u}\bar{u}hW. \tag{9}$$

The quantity γ depends on the velocity distribution in the layer. If the velocity is uniform, $\gamma = 1$. Other physically reasonable assumptions yield values greater than 1 but less than 2 or so. The theory of this paper does not yield a precise value for γ, but we find that γ occurs in combinations with other equally uncertain constants of the problem, so that it seems pointless to attempt to refine this portion of the argument.

Equation (7) becomes

$$\frac{\gamma}{W}\frac{d}{dx}(\overline{uu}Wh) = \frac{h^2}{2}\frac{d\bar{b}}{dx} - h\Delta b\frac{dh}{dx} - K\bar{u}^2, \tag{10}$$

where we have assumed that γ is constant. We have two additional equations involving the flux of mass and buoyancy across an entire section:

$$\overline{ub}h + \bar{u}_0\bar{b}_0 h_0 = 0, \quad \bar{u}_0 h_0 W + \bar{u}hW = q_f W_h, \tag{11),(12}$$

where $\bar{u}_0$ is the mean horizontal velocity over the cross-section in the lower layer, $q_f W_h$ is the fresh-water influx and W_h is the width at the head of the estuary. We have neglected horizontal diffusion of buoyancy in (11).

Let us now non-dimensionalize using the following quantities:

$$Q = \bar{u}hW/q_f W_h, \quad Q_0 = \bar{u}_0 h_0 W/q_f W_h, \quad \xi = x\sigma/q_f, \quad \eta = h\sigma/q_f, \tag{13}$$

$$D = (\bar{b}_0 - \bar{b})/\bar{b}_0, \quad m = \gamma\sigma^3/\bar{b}_0 q_f, \quad R = W/W_h. \tag{14}$$

Equations (3) and (10)–(12) become

$$Q_0 = 1 - D^{-1}, \quad Q = D^{-1}, \tag{15}$$

$$dQ/d\xi = K_n mR/\gamma\eta D, \tag{16}$$

$$\frac{1}{R}\frac{d}{d\xi}\left(\frac{Q^2}{\eta R}\right) = -\frac{1}{2m}\eta^2\frac{dD}{d\xi} - \frac{K}{\gamma}\frac{Q^2}{R^2\eta^2} - \frac{1}{m}\eta D\frac{d\eta}{d\xi}. \tag{17}$$

We may also write (15)–(17) as

$$Q = D^{-1}, \quad Q_0 = 1 - D^{-1}, \tag{18}$$

$$\frac{\eta}{RQ}\frac{dQ}{d\eta} = \frac{K_n m}{\gamma}\frac{d\xi}{d\eta}, \tag{19}$$

$$\frac{d}{d\eta}\left(\frac{Q^2}{\eta R}\right) = -\frac{K}{mK_n R^2}\frac{Q}{\eta}\frac{dQ}{d\eta} + \frac{R}{2m}\frac{\eta^2}{Q^2}\frac{dQ}{d\eta} - \frac{R}{m}\frac{\eta}{Q}. \tag{20}$$

2. The solution for an estuary of uniform width

The solution of (20) for $R = 1$ satisfying the boundary condition $Q = 1$ at the head of the estuary, where $\eta = \eta_h$, is

$$\frac{\eta}{\eta_h} = \left(\frac{\zeta}{\eta_h^3}\right)^{\frac{1}{3}s}\left(\frac{1 + C^3\eta_h^3}{1 + C^3\zeta}\right)^{\frac{1}{3}(s+1)}, \tag{21}$$

where

$$\zeta = \frac{\eta^3}{Q^3}, \quad s = 1 + \frac{1}{1 + K/mK_n}, \quad C^3 = \frac{1/2m}{1 + K/mK_n}. \tag{22}$$

The quantity ξ may be found by integrating (19). We get

$$\xi = \frac{\gamma}{mK_n}\int_{Q_c}^{Q}\frac{\eta}{Q}\,dQ, \tag{23}$$

where we choose Q_c to be the flux at an arbitrary section at which $\xi = 0$.

We may simplify the problem considerably by the new definitions

$$\zeta' = C^3\zeta, \quad \eta' = CQ'_h\eta, \quad Q' = Q'_hQ, \quad \xi' = (mK_n/\gamma)\,CQ'_h\xi, \tag{24}$$

where we denote by Q'_h the value of Q' at the head of the estuary. We then obtain

$$\eta' = (\zeta'^{\frac{1}{3}s}/(1+\zeta')^{\frac{1}{3}(s+1)}, \quad \zeta' = \eta'^3/Q'^3, \tag{25}$$

$$\xi' = \int_{Q_c'}^{Q'} \frac{\eta'}{Q'}\,dQ', \quad Q'_h = \frac{C^{s-1}\eta_h^{s-1}}{(1+C^3\eta_h^3)^{\frac{1}{3}(s+1)}}, \tag{26a, b}$$

where $Q'_c = Q'_h Q_c$. Notice that $1 < s < 2$, tending to $s = 1$ as σ decreases to zero and to $s = 2$ when σ is very large.

We may solve (25) for Q':

$$Q' = (\zeta')^{\frac{1}{3}(s-1)}/(1+\zeta')^{\frac{1}{3}(s+1)}. \tag{27}$$

Differentiating (27), we obtain

$$\frac{dQ'}{d\zeta'} = \frac{\frac{2}{3}[\frac{1}{2}(s-1)-\zeta']\,(\zeta')^{\frac{1}{3}(s-4)}}{(1+\zeta')^{\frac{1}{3}(s+4)}}, \tag{28}$$

so that Q' has a maximum, which we may call Q'_c, when

$$\zeta' = \zeta'_c = \tfrac{1}{2}(s-1). \tag{29}$$

Notice also that

$$\zeta'^{\frac{1}{3}}dQ' = d\xi', \tag{30}$$

so that ξ' and Q' increase or decrease together.

It is enlightening to compute the Froude number F at ζ'_c. A convenient definition is

$$F^2 = \overline{u^2}/h\Delta b. \tag{31}$$

This yields

$$F^2 = \gamma\bar{u}^2/h\Delta b = m/\zeta. \tag{32}$$

Using (24) and (29), we get

$$F_c^2 = 1, \tag{33}$$

so that the maximum flux Q_c corresponds to a point of critical flow with supercritical flow for $\zeta' < \zeta'_c$ and subcritical flow for $\zeta' > \zeta'_c$. Stommel (1951) found a similar behaviour. His theory led to the differential equation (20) with $R = 1$ and the frictional term missing (i.e. $s = 2$). Stommel's theory was incomplete and he could not solve for fluxes and interface depths as functions of distance along the estuary.

Obviously, increasing flux is associated with increasing distance along the estuary towards the mouth. When the flux is a maximum, $\xi' = \xi'_c = 0$ is also a maximum, as indicated by (30). From (25) we get

$$d\eta'/d\zeta' = \tfrac{1}{3}(\zeta')^{\frac{1}{3}(s-3)}(s-\zeta')/(1+\zeta')^{\frac{1}{3}(s+4)}. \tag{34}$$

Let us now consider the solution from the viewpoint of figure 2, which anticipates the argument below that the mouth is at the point $\xi' = \xi'_c = 0$. If we *increase* ζ' from ζ'_c (corresponding to subcritical flow), $s-\zeta' > 0$ and η' increases. It reaches a maximum at $\zeta' = s$ and then decreases as ζ' gets larger. Since Q' is decreasing,

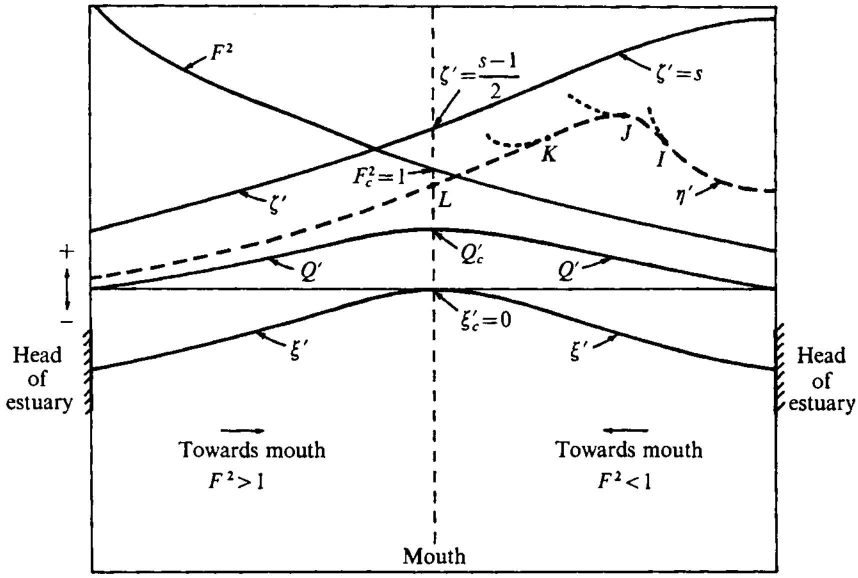

FIGURE 2. Schematic representation of the two solutions.

(30) shows that ξ' is decreasing, so that *we are moving towards the head of the estuary*. On the other hand, if we decrease ζ' from ζ'_c (corresponding to supercritical flow), we find that η' decreases monotonically. Again ξ' decreases, so that we are again moving towards the head of the estuary. Thus two distinct solutions are possible, but the second corresponds to high velocities of a metre per second or more at the head of the estuary and elsewhere and seems unlikely to occur. It also involves a decrease in velocity and an increase in elevation of the free surface towards the mouth, both of which are contrary to observations (Gade 1970). Consequently, we adopt the solution ($\zeta' > \zeta'_c$) for subcritical flow.

The remaining problem concerns the location of the mouth of the estuary. The numerical values of various quantities may be determined as follows: we first specify a value of s between 1 and 2. This determines ζ'_c, the maximum Q'_c and the corresponding value of η'_c. We set $\xi' = 0$ at this section of the channel. We then increase ζ' and determine the corresponding values of Q' and η' from (25) and of ξ' from (26*a*). We may continue the integration until Q' equals Q'_h, but in fact, η_h in (26*b*) is unknown and therefore Q'_h is unknown. Further argument or information is needed to determine uniquely the solution of the problem and it is clear that the effect of the variation of W (which we allowed to vary in the arguments of § 1) must be relevant. Accordingly, let us derive two equations from (16) and (17) yielding the rate of change of η and of $F^2 = mQ^3/\eta^3R^2$ along the channel. We get

$$\frac{d\eta}{d\xi} = \frac{1}{1-F^2}\left\{\frac{2K_n mR}{\gamma}\left(\tfrac{1}{4}-\frac{KF^2}{2mK_nR}-F^2\right)+\frac{\eta F^2}{R}\frac{dR}{d\xi}\right\}, \tag{35}$$

$$\frac{dF^2}{d\xi} = \frac{3F^2K_nmR}{\gamma\eta(1-F^2)}\left[F^2+\tfrac{1}{2}+\frac{KF^2}{mK_nR}\right]-\frac{F^2(F^2+2)}{R(1-F^2)}\frac{dR}{d\xi}. \tag{36}$$

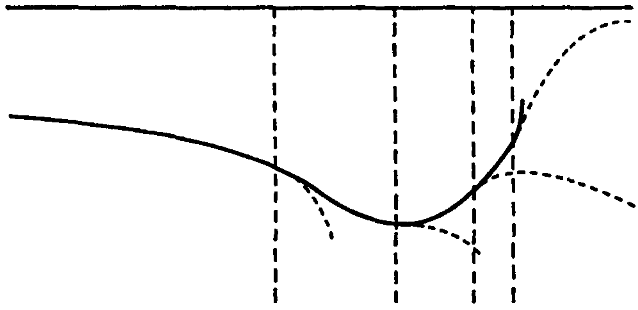

FIGURE 3. Interface variations at the end of the estuary.

Let us now investigate all possibilities, shown in figure 3, for the location of the mouth of the estuary, which we may take to be a section beyond which the width of the estuary increases rapidly.

(*a*) *The estuary ends at or before the section of maximum depth of the upper layer (point I or J).* In this case $1-F^2 > 0$ and the quantity in curly brackets in (35) is positive to the left of I or J, so that η is increasing with ξ. Equation (36) reveals that F^2 is increasing but is considerably less than one. Just past point I or J, where R is still close to one but $dR/d\xi$ is very large, the last term in curly brackets in (35) will quickly dominate and η will increase even more rapidly. At the same time, the last negative term in (36) will dominate and F^2 will start decreasing. If we now regard as physically necessary that η should decrease past the mouth as the mass of brackish water spreads in all directions, this case is impossible.

(*b*) *The estuary ends after the section of maximum depth of the upper layer but not close to the section at which* $F^2 = 1$. In this case, η is decreasing just to the left of point K and F^2 is somewhat smaller than one and increasing. Just past K, however, the last term in curly brackets in (35) quickly dominates and η will reach a minimum and then increase. Also, the last negative term in (36) begins to dominate and F^2 begins to decrease. Again it is not possible for η to decrease past the mouth and this case is also impossible.

(*c*) *The estuary ends just before the section at which* $F^2 = 1$. In this case η is decreasing to the left of point L and F^2 is very close to one. Then, if $d\eta/d\xi$ remains finite, the quantity in curly brackets in (35) falls to zero at a section just past point L, F^2 passes through the critical value and $1-F^2$ becomes negative. After this the last term in curly brackets in (35) dominates and η *continues to decrease.* Equation (36) shows that F^2 also continues to increase. This behaviour is entirely reasonable and we therefore accept the point L as the end of the estuary.

3. Discussion

We see from the theory that $\eta' = g(\xi', s)$, or from (24), $C\eta Q_h' = g(\xi', s)$. Also $Q' = f(\xi', s)$. We obtain then $Q_h' = f(\xi_h', s)$, $Q_c' = f(0, s)$ and

$$C\eta_h = g(\xi_h', s)/f(\xi_h', s), \quad Q_c = f(0, s)/f(\xi_h', s),$$

where ξ_h' is the value of ξ' at the head of the estuary, where $|x| = L$. Computations

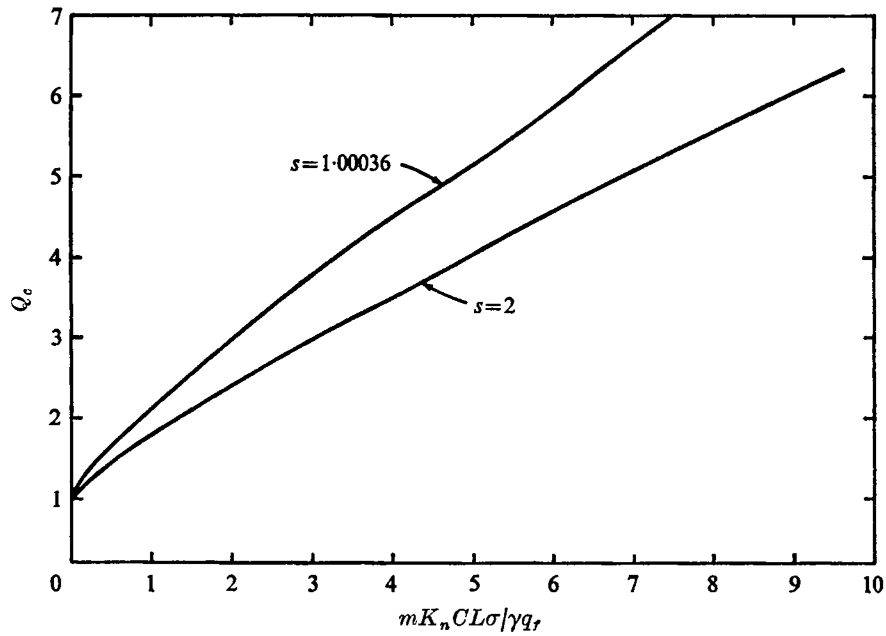

FIGURE 4. Relationship between the mass flux in the upper layer and the estuary length.

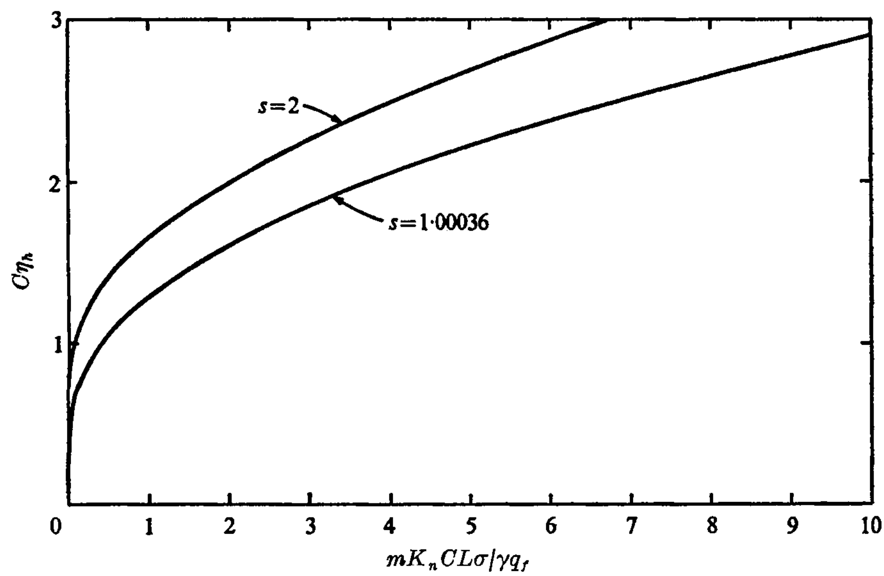

FIGURE 5. Relationship between the halocline depth at the head of the estuary and the estuary length.

for $s = 2$ and $s = 1{\cdot}00036$ are shown in figures 4 and 5. They reveal that the maximum flux and the depth of the interface at the head increase monotonically with the length of the estuary. Other physical interpretations are obscured by the complicated scaling of the non-dimensional variables and we must assign values to the various constants to obtain useful interpretations of the theory.

We may refer to two extreme cases with respect to fresh-water influx, namely the inner Oslofjord (Gade 1970) and Knight Inlet (Pickard & Rodgers 1959), for for which we use $q_f = 60$ and $2{\cdot}3 \times 10^3$ respectively.† It is important, of course, to estimate m in (14), but this quantity is extremely sensitive to the value of σ. In either case, however, and in most estuaries, it seems likely that m is considerably less than 1. The quantity K_n has been estimated as $K_n \cong 0{\cdot}1$. The constant K is uncertain. In flows in channels and pipes, K is very small because the eddy velocities are small compared with mean velocities. This is not true in the present problem, in which σ and $\overline{u}$ are of the same order. An uncertain but reasonable estimate, therefore, is $K = 1$. The estimates we have made so far mean that s is close to but a little larger than 1. The computations from the theory reveal little variation in Q_0 as $s \to 1$. At $mK_nCL\sigma/q_f\gamma = 3$, for example, Q_c increases by about 10% as s decreases from the rather large value of 1·2 to $s = 1{\cdot}00036$. According to figure 4, we may write

$$Q_c \cong 1 + 0{\cdot}9mK_nCL\sigma/\gamma q_f. \tag{37}$$

Our estimates lead to $C = 0{\cdot}37$, and with $\gamma = 1{\cdot}3$, (37) may be written as

$$Q_c \cong 1 + 0{\cdot}033L\sigma^4/\overline{b}_0 q_f^2. \tag{38}$$

In Oslofjord, with $L \cong 2{\cdot}5 \times 10^6$, $\overline{b}_0 \cong 5$ and $Q_c \cong 3{\cdot}2$ (Gade 1970), we estimate $\sigma \cong 0{\cdot}83$. In Knight Inlet, Q_c is quite uncertain, but the data suggest a value of 4. Using $L \cong 1{\cdot}1 \times 10^7$ and $\overline{b}_0 = 25$, we obtain $\sigma = 5{\cdot}7$.

Computations for various values of s near 1 reveal little variation of $C\eta_h$ with s for all but very short estuaries. A rough relationship is

$$C\eta_h \cong 1 + 0{\cdot}25mK_nCL\sigma/\gamma q_f, \tag{39}$$

or approximately,

$$h_h \cong 2{\cdot}7q_f/\sigma + 0{\cdot}025L\sigma^3/\overline{b}_0 q_f. \tag{40}$$

Computations in figures 4 and 5 yield $C\eta_h = 1{\cdot}7$ and $1{\cdot}9$, respectively, for Oslofjord and Knight Inlet. We get approximately $h_h = 3{\cdot}3$ m for Oslofjord and 21 m for Knight Inlet. The first is a considerable underestimate for Oslofjord but the second is close to observations in Knight Inlet. The comparisons suggest that the model of this paper may contain the basic physical mechanisms of Knight Inlet but probably differs fundamentally from Oslofjord. This is not surprising. The former has a geometry similar to the model whereas the latter is very different. For example, Olsofjord has a shallow sill depth, which forms a considerable barrier for the influx of salt water, whereas the sill depths of Knight Inlet are well below the halocline. There is, moreover, an observed basic difference in the horizontal density variation. This is similar to that in the model in Knight Inlet but is virtually absent in Oslofjord, so that the basic driving mechanism of the present model is absent in the latter.

Notice that, if we consider h_h as a function of q_f, (40) shows that the halocline depth is a minimum for a certain value of q_f and large for both small and large values of the fresh-water discharge. This behaviour has been observed in Alberni

† In this section all dimensional quantities are expressed in c.g.s. units.

Inlet by Tully (1949) and in experiments by Welander (1974) and has been discussed at some length by Long (1975*b*).

An interesting feature of the theory is the slope of the free surface. As we have noted, the surface slopes downwards towards the mouth in subcritical flow and (6) yields for the total drop ΔH

$$\Delta H = (q_f \bar{b}_0/\sigma g C)\,(\Delta\zeta')^{\frac{1}{3}}. \tag{41}$$

For Knight Inlet, for example, $(\Delta\zeta')^{\frac{1}{3}} \simeq 2$, and we get $\Delta H \simeq 55\,\mathrm{cm}$. This seems rather large but values of 10–15 cm have been found in shorter Norwegian fjords by Gade (private communication). Notice that the commonly used argument that the free surface *must* slope downwards towards the mouth (Gade 1974, unpublished manuscript) implicitly assumes subcritical flow. We have not discussed the supercritical case, in which the free surface slopes upwards towards the sea, but it is possible that some estuaries have this character.

The calculated drop in surface level for Knight Inlet represents a potential energy far in excess of the observed or theoretical kinetic energies and we conclude that friction dominates the flow. This is probably true of most fjord-type estuaries. We also deduce the dominant effect of friction from the fact that s tends to be very close to 1. As we see in (22), this corresponds to large values of the drag coefficient K. Indeed, the ratio of inertial forces to pressure or frictional forces is proportional to $s-1$.

I wish to thank Anton Edwards of the Dunstaffnage Marine Research Laboratory, Oban, Scotland, Herman Gade, Geophysical Institute, University of Bergen, and Peter Larsen, University of Lund, for the opportunity to visit their institutions and to discuss the work in this paper. This research was supported by the Office of Naval Research, Fluid Dynamics Division, under Contract N00014-75-C-0805, Code N66019.

REFERENCES

Dyer, K. R. 1973 *Estuaries, a Physical Introduction*. Wiley.

Gade, H. 1970 *Hydrographic Investigations in the Oslofjord, a Study of Water Circulation and Exchange Processes*, vols. I–III. Geophys. Inst., University of Bergen, Norway.

Kato, H. & Phillips, O. M. 1969 On the penetration of a turbulent layer into a stratified fluid. *J. Fluid Mech.* **37**, 643–655.

Long, R. R. 1975*a* The influence of shear on mixing across density interfaces. *J. Fluid Mech.* **70**, 305–320.

Long, R. R. 1975*b* On the depth of the halocline in an estuary. *J. Phys. Oceanogr.* **5**, 551–557.

Moore, M. J. & Long, R. R. 1971 An experimental investigation of turbulent stratified shearing flow. *J. Fluid Mech.* **49**, 635–655.

Pickard, G. L. & Rodgers, K. 1959 Current measurements in Knight Inlet, British Columbia. *Fish. Res. Bd Can.* **16**, 635–678.

Pritchard, D. W. 1952 Estuarine hydrography. *Adv. in Geophys.* **1**, 243–280.

Rattray, M. & Hansen, D. V. 1962 A similarity solution for circulation in an estuary. *J. Mar. Res.* **20**, 121–133.

Rouse, H. & Dodu, J. 1955 Turbulent diffusion across a density discontinuity. *Houille Blanche*, **10**, 405–410.

SCHLICHTING, H. 1955 *Boundary Layer Theory*. McGraw-Hill.

STOMMEL, H. 1951 Recent developments in the study of tidal estuaries. *Tech. Rep. W.H.O.I.* no. 51–33.

TULLY, J. P. 1949 Oceanography and prediction of pulp mill pollution in Alberni Inlet. *Bull. Fish. Res. Bd Can.* **83**, 1–169.

TURNER, J. S. 1968 The influence of molecular diffusivity on turbulent entrainment across a density interface. *J. Fluid Mech.* **33**, 639–656.

WELANDER, P. 1974 Two-layer exchange in an estuary basin with special reference to the Baltic Sea. *J. Phys. Oceanog.* **4**, 542–556.